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1996 J. Phys. A: Math. Gen. 29 4977

(http://iopscience.iop.org/0305-4470/29/16/019)

J. Phys. A: Math. Gen. **29** (1996) 4977–4985. Printed in the UK

Topology of energy surfaces and existence of transversal Poincaré sections

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Received 4 March 1996

Abstract. Two questions on the topology of compact energy surfaces of natural two degrees of freedom Hamiltonian systems in a magnetic field are discussed. We show that the topology of this 3-manifold (if it is not a unit tangent bundle) is uniquely determined by the Euler characteristic of the accessible region in configuration space. In this class of 3-manifolds for most cases there does not exist a transverse and complete Poincaré section. We show that there are topological obstacles for its existence such that only in the cases of $S^1 \times S^2$ and T^3 such a Poincaré section can exist.

1. Introduction

The question of the topology of the energy surface of Hamiltonian systems was already treated in the 1920s by Birkhoff [1] and Hotelling [2, 3]. Birkhoff proposed the 'streamline analogy' [1], i.e. the idea that the flow of a Hamiltonian system on the energy surface could be viewed as the streamlines of an incompressible fluid evolving in this manifold. Extending the work of Poincaré [4] he noted that it might be difficult to find a transverse Poincaré section which is complete (i.e. for which every streamline starting from the surface of section returns to it) [5]. Hotelling classified some of the topologies of energy surfaces with two degrees of freedom. In 1970 Smale [6] initiated the study of 'topology and mechanics' from the modern point of view. This work had a great influence and stimulated a lot of research especially in the Russian school of mathematics, see e.g. [7–12].

We want to take the present knowledge about the topology of energy surfaces of natural Hamiltonian systems and return to the question of Birkhoff about the existence of transverse and complete Poincaré surfaces of section. The list of topologies of natural Hamiltonian systems is in principle known, but here we collect the results we need and give a proof using Heegard splittings which explicitly constructs an embedding of the split halves of our 'manifold of streamlines' into $\mathbb{R}^3$. With the help of the computer it is possible to create a realistic picture of Birkhoff's 'streamline analogy' using our result. In the second part the list of topologies of energy surfaces is compared to the list of manifolds that can have a complete and transverse Poincaré section, i.e. which admit the structure of a bundle over S^1 with a Riemann surface as a fibre. In [13] we noted that there can be topological obstacles for the existence of a transverse and complete Poincaré section. We now show that in the class of all energy surfaces of natural Hamiltonian systems (possibly with a magnetic field)

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there can only exist a transverse and complete Poincaré section if the energy surface is a direct product of S^2 or T^2 with S^1.

2. Topology of energy surfaces

Consider a time-independent Hamiltonian system with two degrees of freedom, possibly in a magnetic field, where the kinetic energy is a positive definite quadratic form in the velocities. These Hamiltonians will be called natural in the following. The smooth and orientable two-dimensional configuration space is denoted by Q. The system is described by the Lagrangian on the tangent bundle TQ given by

$$L(q,\dot{q}) = \tfrac{1}{2}\langle \dot{q}, T(q)\dot{q}\rangle - V(q) + \langle A(q), \dot{q}\rangle \tag{1}$$

with a positive definite matrix $T(q)$, potential $V(q)$ and vector potential $A(q)$, where $\langle , \rangle$ denotes the Euclidean standard scalar product. Since $\det T \neq 0$ the momenta are $p = \partial L/\partial \dot{q}$ and the Legendre transformation to T^*Q gives the Hamiltonian

$$H(q,p) = \tfrac{1}{2}\langle (p - A(q)), T^{-1}(q)(p - A(q))\rangle + V(q). \tag{2}$$

The accessible region Q_h in Q for fixed energy $H = h$ is the set of points in Q for which the potential energy does not exceed the total energy

$$Q_h = \{q \in Q \mid V(q) \leqslant h\} \tag{3}$$

which we assume to be compact. Each connected component of Q_h can be treated separately. The ovals of zero velocity with $\dot{q} = 0$ or equivalently $V(q) = h$ are the boundaries of Q_h, if any. The number of ovals of zero velocity, i.e. the number of disjoint components of ∂Q_h is denoted by d. By abuse of language we denote the parts of Q which are excluded from Q_h by the ovals of zero velocity as 'holes' in Q. In the following we always assume that h is a regular value of $H(q, p)$, such that the energy surface

$$\mathcal{E}_h = \{(q,p) \in T^*Q \mid H(q,p) = h\} \tag{4}$$

is smooth. Moreover it is compact because Q_h is assumed to be compact. Note that Q_h is the projection of $\mathcal{E}_h$ onto Q. If Q is compact it is a Riemann surface R^2_g whose genus we denote by g. For a given Q_h with $d > 0$ there are infinitely many compact Q that realize this Q_h because we might attach arbitrarily complicated surfaces to the boundary of the hole. For example, $Q_h \simeq D^2$ is realized by any compact Q if the potential is a Morse function and the energy is sufficiently low. In order to remove this arbitrariness we always think of the holes as to be filled with disks D^2. Our arguments do not depend on this, because they are based on Q_h, and not on Q. Most simple examples of Hamiltonian systems have a non-compact configuration space, in particular $\mathbb{R}^2$ or $S^1 \times \mathbb{R}$. In these cases there must be ovals of zero velocity in order to make Q_h (and thus $\mathcal{E}_h$) compact. Filling these holes with D^2 we obtain a compact Q, such that these cases are included in our treatement.

The case of $d = 0$, i.e. the motion on a compact Riemann surface $Q = R^2_g$ (with sufficiently high energy $h > V(q)$ everywhere) almost by definition (4) has an energy surface homeomorphic to the unit tangent bundle of R^2_g. Here we want to classify all the other cases with $d > 0$.

Proposition 1. The topology of the energy surface $\mathcal{E}_h$ of a natural two degree of freedom Hamiltonian system with compact accessible region of configuration space Q_h is determined by the Euler characteristic χ of Q_h if there is at least one oval of zero velocity.

Proof. Our proof is elementary and constructive: we embed Q in $\mathbb{R}^3$ and attach ellipses of possible velocity to every point of Q_h. Cutting these velocity ellipses we obtain a Heegard splitting of $\mathcal{E}_h$ from which the topology of $\mathcal{E}_h$ is determined.

Since Q is an orientable Riemann surface it can be embedded in $\mathbb{R}^3$:

$$Q \simeq \{r \in \mathbb{R}^3 \mid F(r) = 0\}. \tag{5}$$

In the Lagrangian (1) we now choose r as global coordinates with the additional constraint $F(r) = 0$. The energy function $E(q, \dot{q})$ on TQ is given by

$$\tilde{E}(q, \dot{q}) = \tfrac{1}{2}\langle \dot{q}, T(q)\dot{q}\rangle + V(q) \tag{6}$$

and similarly

$$\tilde{E}(r, \dot{r}) = \tfrac{1}{2}\langle \dot{r}, \tilde{T}(r)\dot{r}\rangle + \tilde{V}(r) \qquad \langle F_r, \dot{r}\rangle = 0 \tag{7}$$

where $\tilde{T}|_Q = T(q)$ and $\tilde{V}|_Q = V(q)$ and the tildes are omitted in the following. The reason for treating everything on TQ instead of T^*Q, is that the linear terms in the momenta in the Hamiltonian due to the vector potential A, are not present if the energy is treated as a function of the velocities $\dot{q}$. Moreover, note that with non-vanishing A on the boundary of Q_h we have zero velocity $\dot{q}$ but not zero momentum p.

With $E(r, \dot{r})$ we have an embedding of $\mathcal{E}_h$ into Euclidean space $\mathbb{R}^6$ given by

$$\mathcal{E}_h \simeq \{(r, \dot{r}) \in \mathbb{R}^6 \mid E(r, \dot{r}) = h, F(r) = 0, \langle F_r, \dot{r}\rangle = 0\}. \tag{8}$$

Following Birkhoff, Hotelling and Smale [1, 2, 6, 14] the energy surface is constructed by attaching circles in velocity space to every point in the (accessible) configuration space Q_h. This gives a fibre bundle with base Q_h and fibre S^1 where the fibre is contracted to a point on ∂Q_h. In our embedding this means to take any point r on $Q \subset \mathbb{R}^3$ and to calculate the remaining kinetic energy $h - V(r)$. Outside Q_h it is negative, on the boundary it is zero and inside of Q_h it is positive. In the latter case the possible velocities are given by $\langle \dot{r}, T\dot{r}\rangle = 2(h - V(r))$. The constraint ensures that $\dot{r}$ is in the tangent plane of $F(r) = 0$ at r. Therefore the possible velocities are located on an ellipse in the tangent plane.

In order to cut the velocity ellipses at every point we need a device to fix a zero position on this S^1, i.e. we want to construct a global section for the fibre bundle. This global section can be constructed with the help of a nowhere vanishing vector field ξ on Q_h. On a Riemann surface Q of genus $g \neq 1$ there does not exist a vector field ξ without equilibrium points. If, however, there are holes (or punctures) in the Riemann surface we can construct ξ on it, such that the restriction to Q_h is without singularities, essentially by moving the singularities into the hole(s). Note that at this point the assumption $d > 0$ is necessary (except for the case of $Q = T^2$). Let $\xi(r)$ be specified in the embedding in $\mathbb{R}^3$ such that $\langle \xi(r), F_r\rangle = 0$. Denote by $N(r)$ the normal vector of the surface $F(r) = 0$. Using $\xi(r)$ every non-zero velocity ellipse can be cut into two halves specified by $\langle N(r), \xi(r) \times \dot{r}\rangle \geqslant 0$ and $\langle N(r), \xi(r) \times \dot{r}\rangle \leqslant 0$, the two halves joining at the place where ξ and $\dot{r}$ are (anti)-parallel. In this way we cut $\mathcal{E}_h$ into two topological equivalent pieces

$$\mathcal{E}_h^{\pm} = \{(r, \dot{r}) \in \mathcal{E}_h \mid \pm \langle N(r), \xi(r) \times \dot{r}\rangle \geqslant 0\} \tag{9}$$

such that

$$\mathcal{E}_h = \mathcal{E}_h^+ \cup \mathcal{E}_h^- \quad \text{and} \quad \partial\mathcal{E}_h{}^+ = \partial\mathcal{E}_h{}^- = \mathcal{E}_h^+ \cap \mathcal{E}_h^-. \tag{10}$$

The two pieces can be embedded into $\mathbb{R}^3$ in the following way. Each half of the velocity ellipse is parametrized by the scalar product $\langle \xi(r), \dot{r}\rangle$. The embedding is defined by

$$\begin{aligned} M : \mathcal{E}_h^{\pm} &\to \mathbb{R}^3 \\ (r, \dot{r}) &\mapsto r + \alpha N(r)\langle \xi(r), \dot{r}\rangle \end{aligned} \tag{11}$$

where α is a sufficiently small constant in order for M to be a homeomorphism. The two solid handle-bodies $M(\mathcal{E}_h^+)$ and $M(\mathcal{E}_h^-)$ coincide in $\mathbb{R}^3$ and define a Heegard splitting of $\mathcal{E}_h$ [15–17]. Their boundaries $M(\partial\mathcal{E}_h{}^{\pm})$ have to be identified to re-obtain $\mathcal{E}_h$. The most important point is that the gluing homeomorphism from $M(\partial\mathcal{E}_h{}^+)$ to $M(\partial\mathcal{E}_h{}^-)$ is the identity map, as is obvious from our construction. The topology of $\mathcal{E}_h$ is therefore completely determined by the topology of $M(\mathcal{E}_h^{\pm})$, which in turn is determined by its boundary $\partial M(\mathcal{E}_h^{\pm}) = M(\partial\mathcal{E}_h{}^{\pm}) = \mathcal{B}$.

The solid handle-body $M(\mathcal{E}_h^{\pm})$ can be thought of as a 'thickened' Q_h because it is obtained by attaching small intervals in the direction of the normal to every interior point, while the interval is contracted to a point on the ovals of zero velocity ∂Q_h. The boundary $\mathcal{B}$ of the solid handle-body is obtained by deleting all the interior points of the attached intervals. The resulting Riemann surface is obtained from two copies of Q_h (corresponding to the two endpoints of each interval) glued together along the ovals of zero velocity ∂Q_h. Analogous to the construction of the energy surface as a bundle over Q_h with fibre S^1 we can think of $\mathcal{B}$ as a bundle over Q_h with fibre S^0 (i.e. two points) where the two points are identified on ∂Q_h.

Q_h is determined by the genus of Q and the number of holes d. The Euler characteristic χ of Q_h is $2-2g-d$ because every hole removes one triangle from the triangulation of R_g^2 which decreases χ by one [15]. To calculate $\chi(\mathcal{B})$ we just double $\chi(Q_h)$ because gluing two holes (i.e. triangles) leaves χ unchanged:

$$\chi(\mathcal{B}) = 2\chi(Q_h) = 4 - 4g - 2d = 2 - 2(2g + d - 1) \tag{12}$$

such that we obtain $2g + d - 1 = 1 - \chi(Q_h)$ for the genus of $\mathcal{B}$. This proves that the topology of $\mathcal{E}_h$ is determined by the Euler characteristic χ of Q_h. □

Denote the genus of $\mathcal{B}$ by $b = 2g + d - 1$. For $b = 0, 1$ there is only one possibility for Q_h, namely with $g = 0$ and $d = 1, 2$. But for larger b we obtain a non-trivial equivalence of energy surfaces for systems on different configuration spaces. The first example of non-trivial equivalence is obtained for $g = 0$, $d = 3$, i.e. a sphere with three holes, which topologically gives the same energy surface as for $g = 1$, $d = 1$, i.e. a torus with one hole. The former system can be realized, for example, by certain spinning tops, while the latter occurs in the double pendulum [13]. Note that the Q_h in these examples are not homeomorphic to each other, even though their Euler characteristic is the same. Most notably for the spinning top Q_h can be embedded into $\mathbb{R}^2$ while for the double pendulum this is impossible.

Our next task is to show that if $\chi(Q_h)$ is different for two energy surfaces then they are not homeomorphic and to actually determine the topology of $\mathcal{E}_h$. The result is well known, in principle, because it follows from the Heegard splittings obtained above (see e.g. [15–17]). We nevertheless give an elementary argument for the cases we need.

Proposition 2. Let there be at least one oval of zero velocity in the compact Q_h and denote the Euler characteristic by $\chi(Q_h) = 1 - b$. For $b = 0$ the energy surface $\mathcal{E}_h$ is homeomorphic to S^3. For $b > 0$ $\mathcal{E}_h$ is homeomorphic to the connected sum of b copies of $S^1 \times S^2$.

Proof. We choose the simplest case $g = 0$ which allows for all possible b. Since $d > 0$ we can map the accessible region Q_h of the sphere S^2 to the Euclidean plane. The resulting disc D^2 has $b = d - 1$ holes. An example can be constructed by considering $H = p^2/2 + V(r)$ ($z = 0$, $p_z = 0$), with b distinct points r_i and $V(r) = r^2 + \sum 1/|r - r_i|$.

For all b we have $N(r) = (0, 0, 1)$ and can, for example, take $\xi(r) = (0, 1, 0)$ as a vector field, such that $\langle \xi, \dot{r} \rangle = \dot{y}$. For this special choice of ξ we actually think of $M(\mathcal{E}_h^{\pm})$

as a projection of $\mathcal{E}_h$ into the Euclidean space $(x, y, \dot{y})$, which produces a double cover in the interior because the sign of $\dot{x}$ is lost.

For $b = 0$ Q_h is a disc D^2 and attaching the intervals of allowed $\dot{y}$ at fixed r (corresponding to each half of the ellipse of possible velocities) we obtain a ball $D^3 \simeq M(\mathcal{E}_h^{\pm})$. Gluing two D^3 along their common boundary $S^2 \simeq M(\partial\mathcal{E}_h{}^{\pm})$ gives $S^3 \simeq \mathcal{E}_h$.

If $b = 1$ then Q_h is an annulus. Attaching the intervals of $\dot{y}$ we now obtain a solid torus $D^2 \times S^1$, whose boundary is T^2, i.e. $b = 1$. Gluing two solid tori by the trivial identification along their boundary gives $S^1 \times S^2$. Note that the trivial identification is important for $b > 0$, because taking a different gluing homeomorphism would yield a different 3-manifold, for example, S^3 or $\mathbb{R}P^3$ for $b = 1$.

In the case of $b = 2$ we have Q_h homeomorphic to a disc D^2 with two holes inside. Attaching the intervals we obtain a solid handle-body $M(\mathcal{E}_h^{\pm})$ whose boundary $\mathcal{B}$ has genus $b = 2$. Now we cut this handle-body into two parts separating the two holes, such that we obtain two solid tori $S^1 \times D^2$. The cut is along a disc D^2. Gluing each solid torus to its partner (leaving the D^2 of the cut unidentified) we obtain $S^1 \times S^2$ with a solid ball D^3 removed. The boundary of this D^3 is S^2, which is obtained by gluing the D^2 of the cut to its partner along their boundary. Now we have to restore the cut to obtain $\mathcal{E}_h$, i.e. we have to glue two copies of $S^1 \times S^2$ along the boundary of a D^3 removed from the two copies. This is exactly the operation of the connected sum and we obtain $\mathcal{E}_h \simeq S^1 \times S^2 \# S^1 \times S^2$ (sometimes this manifold is denoted by K^3). For $b > 2$ the same process is repeated b times and we obtain the connected sum of b copies of $S^1 \times S^2$. $\square$

We summarize our results in the following table, where $M \setminus nD^2$ denotes the two-dimensional manifold M with n disks D^2 removed. By a recent result from Kozlov and Ten [18] all of the combinations of Q_h and $\mathcal{E}_h$ listed in the table can even be realized by natural Hamiltonian systems that are completely integrable.

Let us remark that in the cases without ovals of zero velocity the energy surface is the unit tangent bundle. Most notably $Q = Q_h \simeq S^2$ gives $\mathcal{E}_h \simeq \mathbb{R}P^3$ and $Q = Q_h \simeq T^2$ gives $\mathcal{E}_h \simeq T^3$. All possible $\mathcal{E}_h$ can be realized as mechanical systems. However, most often one encounters S^3, $S^1 \times S^2$, $\mathbb{R}P^3$ and T^3. In the dynamics of the spinning top one can find $\#^2 S^1 \times S^2$ and $\#^3 S^1 \times S^2$ [9, 11, 12].

	$\chi(Q_h)$	1	0	-1	$\cdots$	$1-b$
	b	0	1	2	$\cdots$	b
	$\mathcal{E}_h$	S^3	$S^1 \times S^2$	$\#^2 S^1 \times S^2$	$\cdots$	$\#^b S^1 \times S^2$
	$\mathcal{B}$	S^2	T^2	R_2^2	$\cdots$	R_b^2
Q	Q_h					
S^2		$S^2 \setminus D^2$	$S^2 \setminus 2D^2$	$S^2 \setminus 3D^2$	$\cdots$	$S^2 \setminus (b+1)D^2$
$\mathbb{R}^2$		D^2	$D^2 \setminus D^2$	$D^2 \setminus 2D^2$	$\cdots$	$D^2 \setminus bD^2$
$\mathbb{R} \times S^1$		D^2	$D^1 \times S^1$	$D^1 \times S^1 \setminus D^2$	$\cdots$	$D^1 \times S^1 \setminus (b-1)D^2$
T^2				$T^2 \setminus D^2$	$\cdots$	$T^2 \setminus (b-1)D^2$
$\vdots$						$\vdots$
R_g^2						$R_g^2 \setminus (b+1-2g)D^2$
						$\vdots$
						$R_{[b/2]}^2 \setminus D^2$

3. Non-existence of complete transverse Poincaré sections

With the complete list of topologies of compact energy surfaces of natural Hamiltonian systems with two degrees of freedom at hand, we now want to show that in all cases, except $S^1 \times S^2$ and T^3, a complete transverse Poincaré section is impossible. Using the result of the last section we see that the two exceptions are obtained from Q_h that contain S^1 as a trivial factor, i.e. for $Q_h \simeq S^1 \times D^1$ or T^2.

Let the Poincaré section be defined by a smooth function $S(q, p) = 0$ on phase space. The surface of section Σ_h is obtained by restriction to the energy surface,

$$\Sigma_h = \{(q, p) \in T^*Q \mid H(q, p) = h, S(q, p) = 0\}. \tag{13}$$

If there is more than one component each of them can be treated separately. Excluding cases with critical points, Σ_h is a Riemann surface of arbitrary genus embedded in $\mathcal{E}_h$. The equations of motion are given by the Poisson bracket $\dot{F} = \{F, H\}$. The surface of section is transversal to the flow if

$$\dot{S}|_{(q,p)} \neq 0 \quad \text{for all} \quad (q, p) \in \Sigma_h. \tag{14}$$

Then the Poincaré map $P : \Sigma_h \to \Sigma_h$ is defined by

$$(q, p) \in \Sigma_h \mapsto g^{\tau(q,p)}(q, p) \in \Sigma_h \tag{15}$$

where g^t denotes the Hamiltonian flow and $\tau(q, p)$ is the first return time. We assume that the section is transverse and Σ-complete [13], i.e. that every orbit starting on Σ_h returns to Σ_h and therefore τ is finite and P is well defined on all of Σ_h. A Poincaré section with these properties will be called complete and transverse in the following. The Poincaré map P has degree one due to the existence and uniqueness of the solutions of the differential equation which connects pre-image and image by an integral curve. Hence $\mathcal{E}_h$ has the structure of a fibre bundle with base S^1 and fibre Σ_h [17]. Let ϕ be in S^1. For every base point ϕ the fibre consists of all points $g^{\phi\tau(q,p)/2\pi}(q, p)$ such that $(q, p) \in \Sigma_h$ for $\phi = 0$. The converse formulation is that given the Poincaré mapping the energy surface can be obtained by a suspension into a flow which automatically creates the structure of fibre bundle with base S^1 and fibre Σ_h.

In [13] we have shown that for $\mathcal{E}_h \simeq S^3$ the existence of a transverse section is in contradiction with Liouville's preservation of phase space volume. In the following we use different arguments based on the above bundle structure to treat the general case.

Proposition 3. A complete transverse Poincaré section for a natural two degrees of freedom Hamiltonian system with compact Q_h can only exist for energy surfaces homeomorphic to $S^1 \times S^2$ or T^3. If it exists it can only be realized by the trivial bundle.

Proof. Let us assume there exists a complete transverse section in $\mathcal{E}_h$. This implies that $\mathcal{E}_h$ admits the structure of a fibre bundle with base S^1 and fibre Σ_h

$$\mathcal{E}_h \xrightarrow{\Sigma_h} S^1 \tag{16}$$

as already explained. Let us consider the exact homotopy sequence of this bundle

$$\begin{array}{ccccccc} \pi_2(S^1) & \to & \pi_1(\Sigma_h) & \to & \pi_1(\mathcal{E}_h) & \to & \pi_1(S^1) \\ 0 & \to & \pi_1(\Sigma_h) & \to & \pi_1(\mathcal{E}_h) & \to & \mathbb{Z} \end{array} \tag{17}$$

which implies

$$\pi_1(\mathcal{E}_h)/\pi_1(\Sigma_h) = \mathbb{Z} \tag{18}$$

and therefore

$$\pi_1(\Sigma_h) \subset \pi_1(\mathcal{E}_h). \tag{19}$$

If $\mathcal{E}_h$ is a direct product with S^1 this is obviously possible because

$$\pi_1(M \times N) = \pi_1(M) \times \pi_1(N).$$

For the energy surface $S^1 \times S^2$ and T^3 we have the trivial bundles as a possible solution.

For all other energy surfaces of natural system the bundle structure (16) is impossible. We first treat the cases with ovals of zero velocity. Because $\pi_1(S^3) = id$ (19) cannot hold because id does not have a non-trivial subgroup. Therefore the energy surface S^3 does not admit the bundle structure (16) and therefore does not admit a complete transverse section. For all the other cases of energy surfaces where there are ovals of zero velocity in Q_h we have $b \geqslant 1$ and

$$\pi_1(\#^b S^1 \times S^2) = \pi_1(S^2 \times S^1) * \cdots * \pi_1(S^2 \times S^1) = \mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z} \qquad (20)$$

i.e. π_1 is a free group with b generators. Every subgroup of a free group is a free group, so by (19) $\pi_1(\Sigma_h)$ must be a free group. But Σ_h is a Riemann surface R_s^2 of arbitrary genus s and π_1 of any Riemann surface never is a free group for $s > 0$ [17], which gives us a contradiction. For $s = 0$ we have $\pi_1(S^2) = id$ and (18) gives $\pi_1(\mathcal{E}_h) = \mathbb{Z}$, which contradicts (20) because $b \geqslant 2$.

We now turn to energy surface obtained from Q_h without ovals of zero velocity. If $Q = Q_h = S^2$ we have $\mathcal{E}_h \simeq \mathbb{R}\mathrm{P}^3$ and $\pi_1(\mathbb{R}\mathrm{P}^3) = \mathbb{Z}_2$ is a finite group, so that similar to the case of S^3 equation (19) cannot be fulfilled. For $Q = Q_h = T^2$ we have already seen that $\mathcal{E}_h = T^3$ admits a complete transverse section.

The remaining cases are the energy surface obtained from $Q = Q_h = R_g^2$ with $g > 1$, i.e. the corresponding unit tangent bundles. These energy surfaces already carry a bundle structure, but with base R_g^2 and fibre S^1

$$\mathcal{E}_h \xrightarrow{S^1} R_g^2 \qquad g \geqslant 2 \qquad (21)$$

as opposed to the required structure for a complete transverse section in (16) with $\Sigma_h = R_s^2$,

$$\mathcal{E}_h \xrightarrow{R_s^2} S^1 \qquad s \geqslant 0. \qquad (22)$$

Denote each of the unit tangent bundles described by (21) by U and each of the manifolds admitting a complete transverse Poincaré section by P. We now show that $G_u = \pi_1(U)$ and $G_p = \pi_1(P)$ are different for any choice of U and P. The method of proof is inspired by [19].

Let us first treat the cases with $s \geqslant 2$. As usual for any group G denote by $C(G)$ is centre and by $[G, G]$ its commutant. Now U and P are different because

$$[G_p, G_p] \cap C(G_p) = id \qquad (23)$$

but

$$[G_u, G_u] \cap C(G_u) \neq id \qquad (24)$$

contains at least an infinite cyclic group.

In (18) we observed that G_p contains a normal subgroup G_p' isomorphic to $\pi_1(\Sigma_h)$, such that $G_p/G_p' = \mathbb{Z} = \pi_1(S^1)$. In particular the factor group is commutative. Since the commutant is a minimal normal subgroup such that the corresponding factor group is commutative we have $[G_p, G_p] \in G_p'$. But the fundamental group of a Riemann surface has no centre, i.e. $C(G_p') = id$. Therefore, G_p' does not intersect with $C(G_p)$ and so does not $[G_p, G_p]$, because it is a subgroup of G_p' and we obtain (23).

Consider the group G_u now (see, e.g., [17]). It can be represented as the group generated by $a_1, b_1, \ldots, a_g, b_g$ with the following relations.

(i) Let α be the Euler number of the unit tangent bundle U. Then

$$a_1 b_1 a_1^{-1} b_1{-1} \dots a_g b_g a_g^{-1} b_g^{-1} = z^\alpha. \tag{25}$$

In our case α is just the Euler characteristic $\chi = 2 - 2g$.

(ii) z commutes with any element of the group G_u. In particular z^α belongs to $C(G_u)$.

But it is easily seen from the first relation that $z^\alpha \in [G_u, G_u]$. So the intersection $[G_u, G_u] \cap C(G_u)$ contains a least z^α as stated in (24). Therefore we have shown that the fundamental groups are different, so the manifolds U and P are also different.

Now we have to consider the cases $s = 0, 1$, where the surface of section Σ_h is S^2 or T^2. In the case of the torus $\pi_1(P)$ can be generated by three generators a, b, and z with the relations

$$\begin{aligned} &ab = ba \\ &zaz^{-1} = \phi(a) \\ &zbz^{-1} = \phi(b) \end{aligned} \tag{26}$$

where ϕ is some automorphism of the fundamental group of T^2, i.e. the commutative subgroup generated by a and b. It follows from this that the 1-homology group has at least one generator of infinite order and no more than three generators. But the corresponding homology group for U has more than three generators ($g \geqslant 2$). In the case of the sphere as a surface of section the first homology group of P is $\mathbb{Z}$ so the same argument holds as for T^2, which concludes the proof that for the unit tangent bundles of R_g^2 with $g \geqslant 2$ there does not exist a complete transverse section.

Finally we show that for the cases $\mathcal{E}_h = S^1 \times S^2$ and $\mathcal{E}_h = T^3$ where a complete transverse section exists it can only be constructed from the trivial bundle with $\Sigma_h = S^2$, respectively $\Sigma_h = T^2$. Recall that in both cases $\pi_1(\mathcal{E}_h)$ is commutative. Now both manifolds cannot be realized as S^1 bundles with base R_g^2, $g \geqslant 2$, because the homotopy group of this bundle contains the non-commutative homotopy group of the base R_g^2 as a subgroup, see (19). In the case of $\mathcal{E}_h \simeq T^3$ we have $\pi_1(\mathcal{E}_h) = \mathbb{Z}^3$. But as already mentioned the homology group of the $\Sigma_h = S^2$ bundle over S^1 has only one generator, so that the only possibility is with $\Sigma_h = T^2$. Moreover, we must have $\phi = id$ in (26) in order to obtain $\mathbb{Z}^3$. Therefore the Poincaré section must be obtained from the trivial bundle. For $\mathcal{E}_h \simeq S^1 \times S^2$ we have $\pi_1(\mathcal{E}_h) = \mathbb{Z}$. But the homotopy group of the S^1 bundle of $\Sigma_h = T^2$ contains $\pi_1(\Sigma_h) = \mathbb{Z}^2$ as a subgroup so the only possibility is with $\Sigma_h = S^2$. Finally the only orientable S^2 bundle over S^1 is the trivial bundle. □

4. Discussion

The difficulty in establishing a complete transverse section was already noted by Birkhoff [5], who required a coordinate transformation to exist, which globally introduces an angle ϕ in $\mathcal{E}_h$ and moreover that $\dot\phi \neq 0$. We were not dealing with the second requirement here. Instead we have shown that there are topological obstacles for the existence of such an angle in most energy surfaces, *independently* of the dynamics. We established the non-existence of complete transverse sections in most cases. In particular, there never exists a complete transverse section for geodesic flows on compact orientable Riemann surfaces. For the question of existence in the exceptional cases $\mathcal{E}_h \simeq S^1 \times S^2$ or T^3 the dynamical system has to be considered, i.e. Birkhoffs second condition has to be checked. This additional condition makes it difficult to find complete transverse sections even in the two special cases. In [13] we have shown that for time-reversal Hamiltonians transverse sections for which Σ_h has the same symmetry are impossible. The only examples of complete transverse sections we know of (except for a trivial time periodic forcing which we are not considering here), have a strong vector potential $A(q)$ breaking the time-reversal symmetry (of course their Q_h must be a torus or a cylinder) [13]. These considerations have been our motivation to drop the requirement of transversality and instead try to construct Poincaré sections that are complete, see [13]. We suspect that it is impossible to find a transverse and complete Poincaré section for a natural time-reversible Hamiltonian system.

Acknowledgments

We would like to thank A T Fomenko, S V Matveev, A A Oshemkov and P H Richter for useful discussions. We also thank an unknown referee for helpful comments. This work was partially supported by the Russian Foundation for Fundamental Science (project 95-01-01604) and by the Deutsche Forschungsgemeinschaft.

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